Published by Institute of Physics Publishing for SISSA

Received: *June 26, 2006*
Accepted: *July 9, 2006*
Published: *August 16, 2006*

Current algebra of the pure spinor superstring in $AdS_5 \times S_5$

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Abstract: We perform a Hamiltonian analysis of the classical type IIB superstring on $AdS_5 \times S^5$ in the pure spinor approach. Taking the spatial components of the left-invariant (super)currents and the pure spinor ghosts as canonical variables, we compute the classical graded Poisson brackets of the currents and ghosts and identify the first class constraints associated to the local $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ symmetry and the pure spinor condition. We then study the properties of the BRST generators and the Hamiltonian along the constraints. For a natural choice of the the Lagrange multipliers, we show equivalence of the canonical equations of motion with the covariant ones. Finally we briefly discuss the (non) local currents, including the ghost contribution, that generate the global $PSU(2,2|4)$ symmetry and its Yangian extension in the present framework.

Keywords: Superstrings and Heterotic Strings, AdS-CFT Correspondence.

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Contents

1. Introduction

Quantization of the type IIB superstring on $AdS_5 \times S_5$ remains an open challenging problem. Some progress has been achieved through the pure spinor formalism proposed by Berkovits [1−5][1]. In a recent paper [12], quantum consistency was argued by means of algebraic renormalization arguments. Vertex operators for massless excitations have been proposed some time ago [11] and checked to be classically BRST invariant [22]. However, differently to what happens in flat spacetime [13], very little or nothing is known so far about the emission vertices of massive states. This is a sad state of affairs, in view of the holographic correspondence [14−16] and in particular of the remarkable agreement found in [17−19] [2] between the spectrum of single-trace gauge invariant operators in free $\mathcal{N} = 4$ SYM and the spectrum of the type IIB superstring on $AdS_5 \times S_5$ extrapolated to the point of higher-spin symmetry enhancement. As always in physics, the situation

[1]For review of pure spinor formalism in superstring theory, see [6 − 10].
[2]For review, see [20, 21].

should improve by further exploiting the symmetries of the background. Because of the presence of the RR 5-form flux, worldsheet currents are not chirally split as for instance in WZW models. The study of their quantum OPE may not forgo a classical analysis, which presents some subtleties in view of the non-trivial role of the pure spinor ghosts. For this reason, in the present paper, we study the classical algebra encoded in the (graded) Poisson brackets of the left-invariant (super)currents $J^A_\mu = Str(g^{-1}\partial_\mu g T^A)$ and the ghost currents. To this end we resort to a slightly unconventional approach [32, 33] whereby the spatial components of the (super)currents J^A_1, rather than the supercoset representative itself $g \in PSU(2,2|4)/\,\mathrm{SO}(4,1)\times\mathrm{SO}(5)$, are taken as canonical variables. Along the way, we identify the first class constraints generating the local $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ symmetry and the gauge transformations arising from the pure spinor constraints. We explicitly determine the action of the classical BRST generators on the fundamental worldsheet fields and currents. We then show that the BRST generators commute with the Hamiltonian and we also prove that these BRST generators are nihilpotent along the constraints. A similar analysis in the more standard approach with g as canonical variable, has been performed by Chandia for the heterotic string in the pure spinor formulation [24].

The plan of the paper is as follows. In section 2 we recall some basic facts about the pure spinor formulation of the type IIB superstring on $AdS_5\times S_5$. In section 3, after identifying the momenta Π_A conjugate to the spatial components of the left-invariant currents J^A_1, we compute the classical graded Poisson brackets of the currents in a Hamiltonian approach. In section 4, we study the BRST generators and the Hamiltonian of the theory. In section 5, we derive the canonical equations of motion and show they are equivalent to the covariant ones for a natural choice of the Lagrange multipliers. Conservation and nihilpotency of the BRST charge along the constraints are shown in section 6. In section 7 we briefly address the issue of global symmetries and integrability in the classical Hamiltonian approach. Section 8 contains our conclusions and indicates perspectives for future work. Finally there are two appendices. The first collects our notation and some important features of $PSU(2,2|4)$. The second describes an elementary application of the canonical approach presently exploited to the simple case of a free massless boson.

2. Pure spinor superstring in $AdS_5\times S_5$

As shown in [11, 12, 22] the classical action for the manifestly covariant superstring on $AdS_5\times S_5$ takes the form

$$\begin{aligned} S = & -\int d^2x\sqrt{-\eta}\mathrm{Str}\left[\frac{1}{2}\eta^{\mu\nu}\left(J^{(2)}_\mu J^{(2)}_\nu + J^{(1)}_\mu J^{(3)}_\nu + J^{(3)}_\mu J^{(1)}_\nu\right) + \right. \\ & \left. +\frac{\epsilon^{\mu\nu}}{4}\left(J^{(1)}_\mu J^{(3)}_\nu - J^{(3)}_\mu J^{(1)}_\nu\right)\right] \\ & -\int d^2x\sqrt{-\eta}\mathrm{Str}(w_\mu\mathcal{P}^{\mu\nu}\partial_\nu\lambda + \hat{w}_\mu\tilde{\mathcal{P}}^{\mu\nu}\partial_\nu\hat{\lambda} \\ & +N_\mu\mathcal{P}^{\mu\nu}J^{(0)}_\nu + \hat{N}_\mu\tilde{\mathcal{P}}^{\mu\nu}J^{(0)}_\nu - \frac{1}{2}N_\mu\mathcal{P}^{\mu\nu}\hat{N}_\nu - \frac{1}{2}\hat{N}_\mu\tilde{\mathcal{P}}^{\mu\nu}N_\nu)\,, \end{aligned} \tag{2.1}$$

where we have omitted an overall factor[3] $\sqrt{\lambda}/2\pi = \sqrt{g_s N/\pi}$. We have assumed the world-sheet to be a flat two dimensional space-time with the metric $\eta = \mathrm{diag}(-1,1)$ and labeled the world-sheet coordinates as x^μ, with $\mu,\nu = 0,1$. However we also use the notation $x^0 = t$, $x^1 = x$ and $d^2x = dxdt$. We have also introduced the (chiral) 'projectors'

$$\mathcal{P}^{\mu\nu} = \eta^{\mu\nu} - \epsilon^{\mu\nu}\,, \qquad \tilde{\mathcal{P}}^{\mu\nu} = \eta^{\mu\nu} + \epsilon^{\mu\nu}\,, \qquad \epsilon^{\mu\nu} = \frac{\varepsilon^{\mu\nu}}{\sqrt{-\eta}}\,, \tag{2.2}$$

with $\varepsilon^{01} = -\varepsilon^{10} = 1$ and the left-invariant (super)currents and ghost fields

$$\begin{aligned}
&J^{(0)}_\mu = (g^{-1}\partial_\mu g)^{[\underline{cd}]} T_{[\underline{cd}]}\,, \quad J^{(1)}_\mu = (g^{-1}\partial_\mu g)^{\alpha} T_\alpha\,, \\
&J^{(2)}_\mu = (g^{-1}\partial_\mu g)^{\underline{c}} T_{\underline{c}}\,, \quad J^{(3)}_\mu = (g^{-1}\partial_\mu g)^{\hat\alpha} T_{\hat\alpha}\,, \\
&\lambda = \lambda^\alpha T_\alpha\,, \quad w_\mu = w_{\mu\alpha} K^{\alpha\hat\beta} T_{\hat\beta}\,, \quad \hat\lambda = \hat\lambda^{\hat\alpha} T_{\hat\alpha}\,, \quad \hat w_\mu = \hat w_{\mu\hat\alpha} K^{\hat\alpha\alpha} T_\alpha\,, \\
&N_\mu = -\left\{w_\mu, \lambda\right\} = -w_{\mu\beta}\lambda^\alpha \left\{T_{\hat\beta}, T_\alpha\right\} K^{\beta\hat\beta} = -w_{\mu\beta} K^{\beta\hat\beta} f_{\hat\beta\alpha}^{[\underline{cd}]} \lambda^\alpha T_{[\underline{cd}]}\,, \\
&\hat N_\mu = -\left\{\hat w_\mu, \hat\lambda\right\} = -\hat w_{\mu\hat\alpha}\hat\lambda^{\hat\beta} \left\{T_{\hat\beta}, T_\alpha\right\} K^{\hat\alpha\alpha} = -\hat w_{\mu\hat\alpha} K^{\hat\alpha\alpha} f_{\alpha\hat\beta}^{[\underline{cd}]} \hat\lambda^{\hat\beta} T_{[\underline{cd}]}\,,
\end{aligned} \tag{2.3}$$

where T_A are the (super)generators of $psu(2,2|4)$, some of whose properties can be found in appendix A, where we define our notation, and K^{AB} denotes the inverse of the Cartan-Killing metric.

Following Berkovits, the ghost variables λ and $\hat\lambda$ are chosen to satisfy the pure spinor constraints

$$\lambda^\gamma \gamma^{\underline{c}}_{\gamma\beta} \lambda^\beta = 0\,, \quad \hat\lambda^{\hat\gamma} \gamma^{\underline{c}}_{\hat\gamma\hat\beta} \hat\lambda^{\hat\beta} = 0\,. \tag{2.4}$$

These constraints imply invariance of the action under the gauge transformations

$$\begin{aligned}
&\delta w_{\mu\alpha} \mathcal{P}^{\mu 0} = -\Lambda_{\underline{c}} (\gamma^{\underline{c}})_\alpha\,, \quad \delta w_{\mu\alpha} \mathcal{P}^{\mu 1} = -\Lambda_{\underline{c}} (\gamma^{\underline{c}})_\alpha\,, \\
&\delta \hat w_{\mu\hat\alpha} \hat{\mathcal{P}}^{\mu 0} = -\hat\Lambda_{\underline{c}} (\gamma^{\underline{c}})_{\hat\alpha}\,, \quad \delta \hat w_{\mu\hat\alpha} \hat{\mathcal{P}}^{\mu 1} = \hat\Lambda_{\underline{c}} (\gamma^{\underline{c}})_{\hat\alpha}\,.
\end{aligned} \tag{2.5}$$

Although a promising and thus far consistent formulation of superstring theories the origin of the pure spinor approach is not fully understood. Moreover interpreting the pure spinor constraint (2.4) as the generator of local gauge transformations (2.5) involving $(w_\mu, \hat w_\mu)$ suggests that this symmetry should be gauge fixed at the quantum level in some way. There are many proposals as how to deal with the pure spinor constraint [25 – 30] with no definite widely accepted conclusion.

3. Hamiltonian analysis

In this section we are going to perform the Hamiltonian analysis of the action (2.1). Our analysis is based on the approach introduced in [32] and recently used in the context of the GS superstring in $AdS_5 \times S_5$ in [34].

To begin with note that the left-invariant (super) current defined as $J = g^{-1}dg$ satisfies the zero curvature equation

$$dJ + J \wedge J = 0 \tag{3.1}$$

[3] We work in units $2\pi\alpha' = 1$.

JHEP08(2006)030

or explicitly

$$\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0 \ . \tag{3.2}$$

Using this equation we can express the time component of the current J_0 as

$$\partial_1 J_0 + [J_1, J_0] \equiv D_1 J_0 = \partial_0 J_1 \Rightarrow J_0 = D_1^{-1}(\partial_0 J_1) \ . \tag{3.3}$$

where D_1 is defined by the first equality.

Although slightly unfamiliar, it turns out to be very convenient to choose J_1 as a canonical variable and then to define the conjugate momentum as the variation of the action with respect to $\partial_0 J_1$ [33]. If we replace J_0 in the action (2.1) with (3.3) and then perform the variation with respect to $\partial_0 J_1$ we obtain

$$\begin{aligned}\Pi_J &= \Pi^{(0)} + \Pi^{(1)} + \Pi^{(2)} + \Pi^{(3)} = \\ &= -D_1^{-1}\Big(D_1^{-1}(\partial_0 J_1)^{(2)} + D_1^{-1}(\partial_0 J_1)^{(3)} + D_1^{-1}(\partial_0 J_1)^{(1)} \\ &\quad -\frac{1}{2}J_1^{(3)} + \frac{1}{2}J_1^{(1)} - N_\mu \mathcal{P}^{\mu 0} - \hat{N}_\mu \tilde{\mathcal{P}}^{\mu 0}\Big) \ ,\end{aligned} \tag{3.4}$$

where we have used the fact that

$$\int d^2x \mathrm{Str}[(D_1^{-1}G)F] = -\int d^2x \mathrm{Str}[G(D_1^{-1}F)] \ .$$

We can then introduce the equal-time graded Poisson bracket that for two classical observables F, G depending on the phase super-space variables $Z^A \equiv J_1^A, \Pi_A$ is defined as

$$\{F, G\} = (-1)^{|F||A|}\left[\frac{\partial^L F}{\partial Z^A}\frac{\partial^L G}{\partial \Pi_A} - (-1)^{|A|}\frac{\partial^L F}{\partial \Pi_A}\frac{\partial^L G}{\partial Z^A}\right] \ , \tag{3.5}$$

where the superscript L denotes left derivation. For the components $J_1 = J_1^A T_A$, $\Pi_J = \Pi^A T_A = K^{\mathrm{AB}}\Pi_B T_A$, the above PB's read

$$\left\{J_1^A(x), \Pi_B(y)\right\} = (-1)^{|A|}\delta_B^A \delta(x-y) \tag{3.6}$$

or explicitly

$$\begin{aligned}\left\{J_1^{\underline{c}}(x), \Pi_{\underline{d}}(y)\right\} &= \delta_{\underline{d}}^{\underline{c}}\delta(x-y) \ , \\ \left\{J_1^{[\underline{cd}]}(x), \Pi_{[\underline{ef}]}(y)\right\} &= \delta_{[\underline{ef}]}^{[\underline{cd}]}\delta(x-y) \ , \\ \left\{J_1^{\alpha}(x), \Pi_{\beta}(y)\right\} &= -\delta_{\beta}^{\alpha}\delta(x-y) \ , \\ \left\{J_1^{\hat{\alpha}}(x), \Pi_{\hat{\beta}}(y)\right\} &= -\delta_{\hat{\beta}}^{\hat{\alpha}}\delta(x-y) \ .\end{aligned} \tag{3.7}$$

It is convenient to define Π^A as

$$\Pi^A = K^{\mathrm{AB}}\Pi_B \tag{3.8}$$

and to express J_0^A as a function of the canonical variables J_1^A, Π^A. With the help of (3.4) we get

$$\begin{aligned}
J_0^{\alpha} &= -(\partial_1 \Pi^{\alpha} + J_1^{[\underline{cd}]} \Pi^{\beta} f^{\alpha}_{[\underline{cd}]\beta} + J_1^{\beta} \Pi^{[\underline{cd}]} f^{\alpha}_{\beta[\underline{cd}]} + J_1^{\underline{c}} \Pi^{\hat{\alpha}} f^{\alpha}_{\underline{c}\hat{\alpha}} + J_1^{\hat{\alpha}} \Pi^{\underline{c}} f^{\alpha}_{\hat{\alpha}\underline{c}}) - \frac{1}{2} J_1^{\alpha} \, , \\
J_0^{\hat{\alpha}} &= -(\partial_1 \Pi^{\hat{\alpha}} + J_1^{[\underline{cd}]} \Pi^{\hat{\beta}} f^{\hat{\alpha}}_{[\underline{cd}]\hat{\beta}} + J_1^{\hat{\beta}} \Pi^{[\underline{cd}]} f^{\hat{\alpha}}_{\hat{\beta}[\underline{cd}]} + J_1^{\underline{c}} \Pi^{\alpha} f^{\hat{\alpha}}_{\underline{c}\alpha} + J_1^{\alpha} \Pi^{\underline{c}} f^{\hat{\alpha}}_{\alpha\underline{c}}) + \frac{1}{2} J_1^{\hat{\alpha}} \, , \\
J_0^{\underline{c}} &= -(\partial_1 \Pi^{\underline{c}} + J_1^{[\underline{cd}]} \Pi^{\underline{f}} f^{\underline{c}}_{[\underline{cd}]\underline{f}} + J_1^{\underline{f}} \Pi^{[\underline{cd}]} f^{\underline{c}}_{\underline{f}[\underline{cd}]} + J_1^{\alpha} \Pi^{\beta} f^{\underline{c}}_{\alpha\beta} + J_1^{\hat{\alpha}} \Pi^{\hat{\beta}} f^{\underline{c}}_{\hat{\alpha}\hat{\beta}}) \, , \\
\Phi^{[\underline{cd}]} &= \partial_1 \Pi^{[\underline{cd}]} + J_1^{[\underline{ef}]} \Pi^{[\underline{gh}]} f^{[\underline{cd}]}_{[\underline{ef}][\underline{gh}]} + J_1^{\hat{\alpha}} \Pi^{\alpha} f^{[\underline{cd}]}_{\hat{\alpha}\alpha} + J_1^{\alpha} \Pi^{\hat{\beta}} f^{[\underline{cd}]}_{\alpha\hat{\beta}} + J^{\underline{e}} \Pi^{\underline{f}} f^{[\underline{cd}]}_{\underline{ef}} - \\
&\quad - N_{\mu}^{[\underline{cd}]} \mathcal{P}^{\mu 0} - \hat{N}_{\mu}^{[\underline{cd}]} \tilde{\mathcal{P}}^{\mu 0} \, . \qquad (3.9)
\end{aligned}$$

With (3.9) in mind, we observe few important points. Firstly, the expression Φ is the constraint that reflects invariance of the action under local gauge $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ transformations. Secondly, due to the fact that, contrary to the standard GS action, the action (2.1) contains time components of the currents J^{α}, $J^{\hat{\alpha}}$, it is not invariant under local κ symmetry. As a result, in the present approach, the Hamiltonian analysis performed above does not generate the 'troublesome' fermionic constraints of the GS approach that cannot be covariantly split into first and second class, the former being the generators of κ symmetry [34]. Yet, as we will momentarily see, the pure spinor constraint could be viewed as the generator of local gauge transformation of the w and $\hat{w}$ conjugate ghosts.

3.1 Graded Poisson brackets of the currents

In this subsection we determine the graded algebra of Poisson brackets of the currents using (3.7) and also (3.9).

To begin with, note that by definition, the Poisson bracket between currents with spatial components is equal to zero

$$\left\{ J_1^A(x), J_1^B(y) \right\} = 0 \, . \qquad (3.10)$$

Then it is rather straightforward to evaluate the Poisson brackets of $J_0^A(x)$ and $J_1^B(y)$. Using (3.9) and (3.7) we get

$$\left\{ J_0^A(x), J_1^B(y) \right\} = K^{\mathrm{AB}} \partial_x \delta(x-y) + J_1^C(x) f^A_{CD} K^{\mathrm{DB}} \delta(x-y) \, , \qquad (3.11)$$

or more explicitly

$$\begin{aligned}
\left\{ J_0^{\alpha}(x), J_1^{\beta}(y) \right\} &= J_1^{\underline{c}}(x) f^{\alpha}_{\underline{c}\hat{\alpha}} K^{\hat{\alpha}\beta} \delta(x-y) \, , \\
\left\{ J_0^{\alpha}(x), J_1^{\hat{\beta}}(y) \right\} &= K^{\alpha\hat{\beta}} \partial_x \delta(x-y) + J_1^{[\underline{cd}]}(x) f^{\alpha}_{[\underline{cd}]\beta} K^{\beta\hat{\beta}} \delta(x-y) \, , \\
\left\{ J_0^{\alpha}(x), J_1^{\underline{c}}(y) \right\} &= J_1^{\hat{\alpha}}(x) f^{\alpha}_{\hat{\alpha}\underline{d}} K^{\underline{dc}} \delta(x-y) \, , \\
\left\{ J_0^{\alpha}(x), J_1^{[\underline{cd}]}(y) \right\} &= J_1^{\beta}(x) f^{\alpha}_{\beta[\underline{ef}]} K^{[\underline{ef}][\underline{cd}]} \delta(x-y) \, , \\
\left\{ J_0^{\hat{\alpha}}(x), J_1^{\hat{\beta}}(y) \right\} &= J_1^{\underline{c}}(x) f^{\hat{\alpha}}_{\underline{c}\alpha} K^{\alpha\hat{\beta}} \delta(x-y) \, , \\
\left\{ J_0^{\hat{\alpha}}(x), J_1^{\beta}(y) \right\} &= K^{\hat{\alpha}\beta} \partial_x \delta(x-y) + J_1^{[\underline{cd}]}(x) f^{\hat{\alpha}}_{[\underline{cd}]\hat{\beta}} K^{\hat{\beta}\beta} \delta(x-y) \, ,
\end{aligned}$$

$$
\begin{aligned}
\left\{J_0^{\hat\alpha}(x), J_1^{\underline{c}}(y)\right\} &= J_1^{\alpha}(x) f_{\alpha\underline{d}}^{\hat\alpha} K^{\underline{dc}}\delta(x-y)\,,\\
\left\{J_0^{\underline{c}}(x), J_1^{\alpha}(y)\right\} &= J_1^{\hat\alpha}(x) f_{\hat\alpha\hat\beta}^{\underline{c}} K^{\hat\beta\alpha}\delta(x-y)\,,\\
\left\{J_0^{\underline{c}}(x), J_1^{\hat\alpha}(y)\right\} &= J_1^{\alpha}(x) f_{\alpha\beta}^{\underline{c}} K^{\beta\hat\alpha}\delta(x-y)\ ,\\
\left\{J_0^{\underline{c}}(x), J_1^{\underline{d}}(y)\right\} &= K^{\underline{cd}}\partial_x\delta(x-y) + J_1^{[\underline{fg}]}(x) f_{[\underline{fg}]\underline{e}}^{\underline{c}} K^{\underline{ed}}\delta(x-y)\,,\\
\left\{J_0^{\underline{c}}(x), J_1^{[\underline{cd}]}(y)\right\} &= J_1^{\underline{d}}(x) f_{\underline{d}[\underline{ef}]}^{\underline{c}} K^{[\underline{ef}][\underline{cd}]}\delta(x-y)\,,\\
\left\{J_0^{\hat\alpha}(x), J_1^{[\underline{cd}]}(y)\right\} &= J_1^{\hat\beta}(x) f_{\hat\beta[\underline{ef}]}^{\hat\alpha} K^{[\underline{ef}][\underline{cd}]}\delta(x-y)\ .
\end{aligned}
\tag{3.12}
$$

The structure of these PB's deserves some comments. The first important feature to notice is that they are not manifestly covariant w.r.t. two dimensional worldsheet transformations. This is a consequence of the non covariant equal time Hamiltonian formalism. Another feature is the presence of the non ultra-local terms $\partial_x\delta(x-y)$. They arise as a result of our choice of canonical variables. For a better understanding of this approach, we will perform in appendix A a similar analysis in the simplest case of a two dimensional free bosonic theory. Finally and very importantly, the PB's respect the Z_4 grading dictated by the underlying $PSU(2,2|4)$ structure.

Along the same line, we can calculate the Poisson brackets between the constraint Φ and the spatial currents J_1^A

$$
\begin{aligned}
\left\{\Phi^{[\underline{cd}]}(x), J_1^{[\underline{ef}]}(y)\right\} &= -\partial_x\delta(x-y) K^{[\underline{cd}][\underline{ef}]} - J_1^{[\underline{ab}]}(x) f_{[\underline{ab}][\underline{gh}]}^{[\underline{cd}]} K^{[\underline{gh}][\underline{ef}]}\delta(x-y)\ ,\\
\left\{\Phi^{[\underline{gh}]}(x), J_1^{\underline{c}}(y)\right\} &= -J_1^{\underline{d}}(x) f_{\underline{de}}^{[\underline{gh}]} K^{\underline{ec}}\delta(x-y)\,,\\
\left\{\Phi^{[\underline{cd}]}(x), J_1^{\alpha}(y)\right\} &= -J_1^{\gamma}(x) f_{\gamma\hat\beta}^{[\underline{cd}]} K^{\hat\beta\alpha}\delta(x-y)\,,\\
\left\{\Phi^{[\underline{cd}]}(x), J_1^{\hat\alpha}(y)\right\} &= -J_1^{\hat\gamma}(x) f_{\hat\gamma\beta}^{[\underline{cd}]} K^{\beta\hat\alpha}\delta(x-y)\,,
\end{aligned}
\tag{3.13}
$$

that explicitly show how the left-invariant currents transform under the (right) gauge transformations generated by $\Phi^{[\underline{cd}]}$.

More involved is the calculation of the Poisson brackets $\left\{J_0^A(x), J_0^B(y)\right\}$. Again we have to resort on (3.9) and (3.7) as well as on the (anti) symmetry properties

$$
f_{\mathrm{AB}}^{E} K^{\mathrm{BF}} = -(-1)^{|B||C|} f_{\mathrm{AC}}^{D} K^{\mathrm{CE}} K_{\mathrm{DB}} K^{\mathrm{BF}} = -(-1)^{|E||F|} f_{\mathrm{AC}}^{F} K^{\mathrm{CE}}
\tag{3.14}
$$

and the graded Jacobi identities

$$
0 = (-1)^{|A||C|} f_{\mathrm{AD}}^{E} f_{\mathrm{BC}}^{D} + (-1)^{|B||A|} f_{\mathrm{BD}}^{E} f_{\mathrm{CA}}^{D} + (-1)^{|C||B|} f_{\mathrm{CD}}^{E} f_{\mathrm{AB}}^{D} = 0\ .
\tag{3.15}
$$

After straightforward though rather tedious calculations we obtain

$$
\begin{aligned}
\left\{J_0^{\underline{c}}(x), J_0^{\underline{d}}(y)\right\} &= -(\Phi^{[\underline{fg}]} + N_\mu^{[\underline{fg}]}\mathcal{P}^{\mu 0} + \hat N_\mu^{[\underline{fg}]}\tilde{\mathcal{P}}^{\mu 0})(x) f_{[\underline{fg}]\underline{e}}^{\underline{c}} K^{\underline{ed}}\delta(x-y)\,,\\
\left\{J_0^{\underline{c}}(x), J_0^{\alpha}(y)\right\} &= (J_0^{\hat\beta} - J_1^{\hat\beta})(x) f_{\hat\beta\hat\gamma}^{\underline{c}} K^{\hat\gamma\alpha}\delta(x-y)\,,\\
\left\{J_0^{\underline{c}}(x), J_0^{\hat\alpha}(y)\right\} &= (J_0^{\beta} + J_1^{\beta})(x) f_{\beta\gamma}^{\underline{c}} K^{\gamma\hat\alpha}\delta(x-y)\,,\\
\left\{J_0^{\underline{c}}(x), \Phi^{[\underline{de}]}(y)\right\} &= -J_0^{\underline{d}}(x) f_{\underline{d}[\underline{fg}]}^{\underline{c}} K^{[\underline{fg}][\underline{de}]}\delta(x-y)
\end{aligned}
\tag{3.16}
$$

and

$$
\begin{aligned}
\left\{J_0^{\alpha}(x), J_0^{\beta}(y)\right\} &= (J_0^{\underline{c}} - J_1^{\underline{c}})(x) f_{\underline{c}\hat{\alpha}}^{\alpha} K^{\hat{\alpha}\beta} \delta(x-y)\,, \\
\left\{J_0^{\hat{\alpha}}(x), J_0^{\hat{\beta}}(y)\right\} &= (J_0^{\underline{c}} + J_1^{\underline{c}})(x) f_{\underline{c}\alpha}^{\hat{\alpha}} K^{\alpha\hat{\beta}} \delta(x-y)\,, \\
\left\{J_0^{\alpha}(x), J_0^{\hat{\alpha}}(y)\right\} &= -(\Phi^{[\underline{cd}]} + N_{\mu}^{[\underline{cd}]}\mathcal{P}^{\mu 0} + \hat{N}_{\mu}^{[\underline{cd}]}\tilde{\mathcal{P}}^{\mu 0})(x) f_{[\underline{cd}]\gamma}^{\alpha} K^{\gamma\hat{\alpha}} \delta(x-y)\,, \\
\left\{J_0^{\alpha}(x), \Phi^{[\underline{cd}]}(y)\right\} &= -J_0^{\gamma}(x) f_{\gamma[\underline{ef}]}^{\alpha} K^{[\underline{ef}][\underline{cd}]} \delta(x-y)\,, \\
\left\{J_0^{\hat{\alpha}}(x), \Phi^{[\underline{cd}]}(y)\right\} &= -J_0^{\hat{\gamma}}(x) f_{\hat{\gamma}[\underline{ef}]}^{\hat{\alpha}} K^{[\underline{ef}][\underline{cd}]} \delta(x-y)\;.
\end{aligned}
\tag{3.17}
$$

Finally we also need the Poisson brackets of the generators of the gauge transformations

$$
\left\{\Phi^{[\underline{cd}]}(x), \Phi^{[\underline{ef}]}(y)\right\} = -\Phi^{[\underline{ab}]}(x) f_{[\underline{ab}][\underline{gh}]}^{[\underline{cd}]} K^{[\underline{gh}][\underline{ef}]} \delta(x-y)\;. \tag{3.18}
$$

Using the above form of the current algebra, we will momentarily derive the classical Hamiltonian and the field equations, and prove the nihilpotency and conservation of the classical BRST charges. It is also clear that using the Poisson brackets given above we can find the Poisson brackets between the chiral components of the currents $J_{\pm}^{A}$ which are related to J_z^A and $J_{\bar{z}}^A$ after Wick rotation. We will demonstrate a simple instance of this calculation in the next section where we will also calculate the Poisson brackets between the BRST charges and some chiral currents. However due to the non-chirally split structure of the algebra, for our purposes, it is more convenient to work with the Poisson brackets given above.

4. BRST charges and Hamiltonian

In this section, we discuss the Hamiltonian and the BRST charges together with their properties. We then derive the classical canonical equations of motion in the next section.

As a first step, we need the action of the BRST charges on the currents and ghost fields. To begin with we express $N^{[\underline{cd}]}, \hat{N}^{[\underline{cd}]}$ using the ghosts and their conjugate momenta. Since

$$
\begin{aligned}
w_{\mu\alpha}\mathcal{P}^{\mu 0} &= -\pi_{\alpha}\,, \qquad & w_{\mu\alpha}\mathcal{P}^{\mu 1} &= -\pi_{\alpha}\,, \\
\hat{w}_{\mu\hat{\alpha}}\tilde{\mathcal{P}}^{\mu 0} &= -\hat{\pi}_{\hat{\alpha}}\,, \qquad & \hat{w}_{\mu\hat{\alpha}}\tilde{\mathcal{P}}^{\mu 1} &= \hat{\pi}_{\hat{\alpha}}
\end{aligned}
\tag{4.1}
$$

we obtain

$$
\begin{aligned}
N_{\mu}^{[\underline{cd}]}\mathcal{P}^{\mu 0} &= \pi_{\beta} K^{\beta\hat{\beta}} f_{\hat{\beta}\alpha}^{[\underline{cd}]} \lambda^{\alpha} \equiv N^{[\underline{cd}]}\,, \\
N_{\mu}^{[\underline{cd}]}\mathcal{P}^{\mu 1} &= \pi_{\beta} K^{\beta\hat{\beta}} f_{\hat{\beta}\alpha}^{[\underline{cd}]} \lambda^{\alpha} = N^{[\underline{cd}]}\,, \\
\hat{N}_{\mu}^{[\underline{cd}]}\mathcal{P}^{\mu 0} &= \hat{\pi}_{\hat{\beta}} K^{\hat{\beta}\beta} f_{\beta\hat{\gamma}}^{[\underline{cd}]} \hat{\lambda}^{\hat{\gamma}} \equiv \hat{N}^{[\underline{cd}]}\,, \\
\hat{N}_{\mu}^{[\underline{cd}]}\mathcal{P}^{\mu 1} &= -\hat{\pi}_{\hat{\beta}} K^{\hat{\beta}\beta} f_{\beta\hat{\gamma}}^{[\underline{cd}]} \hat{\lambda}^{\hat{\gamma}} = -\hat{N}^{[\underline{cd}]}\,,
\end{aligned}
\tag{4.2}
$$

where π, λ and $\hat{\pi}, \hat{\lambda}$ satisfy the canonical Poisson brackets

$$
\{\lambda^{\alpha}(x), \pi_{\beta}(y)\} = \delta_{\beta}^{\alpha}\delta(x-y)\,, \qquad \left\{\hat{\lambda}^{\hat{\alpha}}(x), \hat{\pi}_{\hat{\beta}}(y)\right\} = \delta_{\hat{\beta}}^{\hat{\alpha}}\delta(x-y)\;. \tag{4.3}
$$

However one has to consider effect of the pure spinor constraints on the system (2.4). Their presence implies that it is natural to study the classical dynamics of the ghosts system as the dynamics of a constrained system. Using (4.1) and (4.3), it is easy to see that the constraints

$$\Phi^{\underline{c}} = \frac{1}{2}\lambda^{\alpha}\gamma^{\underline{c}}_{\alpha\beta}\lambda^{\beta} , \quad \hat{\Phi}^{\underline{c}} = \frac{1}{2}\hat{\lambda}^{\hat{\alpha}}\gamma^{\underline{c}}_{\hat{\alpha}\hat{\beta}}\hat{\lambda}^{\hat{\beta}} . \tag{4.4}$$

generate the gauge transformations (2.5) since

$$\{\Phi^{\underline{c}}(x), \pi_{\beta}(y)\} = \gamma^{\underline{c}}_{\beta\gamma}\lambda^{\gamma}(y)\delta(x-y) , \left\{\hat{\Phi}^{\underline{c}}(x), \hat{\pi}_{\hat{\beta}}(y)\right\} = \gamma^{\underline{c}}_{\hat{\beta}\hat{\gamma}}\hat{\lambda}^{\hat{\gamma}}(y)\delta(x-y) . \tag{4.5}$$

Using (4.5) we also obtain

$$\begin{aligned} \left\{\Phi^{\underline{c}}(x), N^{[\underline{de}]}(y)\right\} &= -\Phi^{\underline{h}}(x) f^{\underline{c}}_{\underline{h}[\underline{ab}]} K^{[\underline{ab}][\underline{de}]}\delta(x-y) , \\ \left\{\hat{\Phi}^{\underline{c}}(x), \hat{N}^{[\underline{de}]}(y)\right\} &= -\hat{\Phi}^{\underline{h}}(x) f^{\underline{c}}_{\underline{h}[\underline{ab}]} K^{[\underline{ab}][\underline{de}]}\delta(x-y) . \end{aligned} \tag{4.6}$$

In the same way we can show that

$$\left\{\Phi^{\underline{c}}(x), \Phi^{\underline{d}}(y)\right\} = 0 , \quad \left\{\hat{\Phi}^{\underline{c}}(x), \hat{\Phi}^{\underline{d}}(y)\right\} = 0 . \tag{4.7}$$

This result implies that the pure spinor constraints are first class.

Let us then consider the Poisson bracket of $\Phi^{[\underline{cd}]}$ with the ghost variables. Using the explicit form of $\Phi^{[\underline{cd}]}$ given in (3.9) and also (4.2) together with (4.3) we obtain

$$\begin{aligned} \left\{\Phi^{[\underline{cd}]}(x), \lambda^{\alpha}(y)\right\} &= -\lambda^{\beta}(x) f^{[\underline{cd}]}_{\beta\hat{\beta}} K^{\hat{\beta}\alpha}\delta(x-y) , \\ \left\{\Phi^{[\underline{cd}]}(x), \hat{\lambda}^{\hat{\alpha}}(y)\right\} &= -\hat{\lambda}^{\hat{\beta}}(x) f^{[\underline{cd}]}_{\hat{\beta}\beta} K^{\beta\hat{\beta}}\delta(x-y) \end{aligned} \tag{4.8}$$

that explicitly demonstrates that λ, $\hat{\lambda}$ transform nontrivially under $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ gauge transformations. Moreover, (4.8) also implies

$$\begin{aligned} \left\{\Phi^{[\underline{cd}]}(x), \Phi^{\underline{e}}(y)\right\} &= \Phi^{\underline{f}}(x) f^{\underline{e}}_{\underline{f}[\underline{ab}]} K^{[\underline{ab}][\underline{cd}]}\delta(x-y) , \\ \left\{\Phi^{[\underline{cd}]}(x), \hat{\Phi}^{\underline{e}}(y)\right\} &= \hat{\Phi}^{\underline{f}}(x) f^{\underline{e}}_{\underline{f}[\underline{ab}]} K^{[\underline{ab}][\underline{cd}]}\delta(x-y) . \end{aligned} \tag{4.9}$$

Then (3.18), (4.7) and (4.9) show that $\Phi^{[\underline{cd}]}, \Phi^{\underline{c}}, \hat{\Phi}^{\underline{c}}$ consist of only first class constraints. This fact will be important below.

For later purposes we here determine the following Poisson brackets

$$\begin{aligned} \left\{\Phi^{[\underline{cd}]}(x), \pi_{\beta}(y)\right\} &= K^{[\underline{cd}][\underline{ef}]} f^{\gamma}_{[\underline{ef}]\beta}\pi_{\gamma}(y)\delta(x-y) , \\ \left\{\Phi^{[\underline{cd}]}(x), \hat{\pi}_{\hat{\beta}}(y)\right\} &= K^{[\underline{cd}][\underline{ef}]} f^{\hat{\gamma}}_{[\underline{ef}]\hat{\beta}}\hat{\pi}_{\hat{\gamma}}(y)\delta(x-y) \end{aligned} \tag{4.10}$$

and

$$\left\{\Phi^{[\underline{cd}]}(x), N^{[\underline{ef}]}(y)\right\} = -N^{[\underline{gh}]}(y) f^{[\underline{cd}]}_{[\underline{gh}][\underline{ab}]} K^{[\underline{ab}][\underline{ef}]}\delta(x-y) , \tag{4.11}$$

$$\left\{\Phi^{[\underline{cd}]}(x), \hat{N}^{[\underline{ef}]}(y)\right\} = -\hat{N}^{[\underline{gh}]}(y) f^{[\underline{cd}]}_{[\underline{gh}][\underline{ab}]} K^{[\underline{ab}][\underline{ef}]}\delta(x-y) . \tag{4.12}$$

4.1 Classical BRST generators

We are ready to study the action of the BRST charges on the fundamental fields that appear in the action (2.1). As shown in [11, 12, 22] the BRST charges take the form

$$
\begin{aligned}
Q_R &= \int dx \hat{\lambda}^{\hat{\alpha}} K_{\hat{\alpha}\alpha} J^{\alpha}_{\mu} \tilde{\mathcal{P}}^{\mu 0} = -\int dx \hat{\lambda}^{\hat{\alpha}} K_{\hat{\alpha}\beta} [J^{\beta}_0 + J^{\beta}_1] \,, \\
Q_L &= \int dx \lambda^{\alpha} K_{\alpha\hat{\beta}} J^{\hat{\beta}}_{\mu} \mathcal{P}^{\mu 0} = -\int dx \lambda^{\alpha} K_{\alpha\hat{\beta}} [J^{\hat{\beta}}_0 - J^{\hat{\beta}}_1] \,.
\end{aligned} \tag{4.13}
$$

Then using the Poisson brackets determined in the previous section we easily get

$$
\begin{aligned}
\{Q_R, J^{\underline{c}}_1(y)\} &= -\hat{\lambda}^{\hat{\alpha}} J^{\hat{\beta}}_1(y) f^{\underline{c}}_{\hat{\alpha}\hat{\beta}} \,, \qquad \{Q_L, J^{\underline{c}}_1(y)\} = -\lambda^{\alpha} J^{\beta}_1(y) f^{\underline{c}}_{\alpha\beta} \,, \\
\{Q_R, J^{\underline{c}}_0(y)\} &= -\hat{\lambda}^{\hat{\alpha}} J^{\hat{\beta}}_0(y) f^{\underline{c}}_{\hat{\alpha}\hat{\beta}} \,, \qquad \{Q_L, J^{\underline{c}}_0(y)\} = -\lambda^{\alpha} J^{\beta}_0(y) f^{\underline{c}}_{\alpha\beta} \,, \\
\{Q_R, J^{\alpha}_1(y)\} &= -\hat{\lambda}^{\hat{\alpha}} J^{\underline{c}}_1(y) f^{\alpha}_{\hat{\alpha}\underline{c}} \,, \qquad \left\{Q_L, J^{\hat{\alpha}}_1(y)\right\} = -\lambda^{\beta} J^{\underline{c}}_1(y) f^{\hat{\alpha}}_{\beta\underline{c}} \,, \\
\{Q_L, J^{\alpha}_1(y)\} &= \partial_1 \lambda^{\alpha}(y) + J^{[\underline{cd}]}_1 \lambda^{\beta}(y) f^{\alpha}_{[\underline{cd}]\beta} \equiv \nabla_1 \lambda^{\alpha}(y) \,, \\
\left\{Q_R, J^{\hat{\alpha}}_1(y)\right\} &= \partial_1 \hat{\lambda}^{\hat{\alpha}}(y) + J^{[\underline{cd}]}_1 \hat{\lambda}^{\hat{\beta}}(y) f^{\hat{\alpha}}_{[\underline{cd}]\hat{\beta}} \equiv \nabla_1 \hat{\lambda}^{\hat{\alpha}}(y)
\end{aligned} \tag{4.14}
$$

where $\nabla_1 X^A = \partial_1 X^A + J^{[\underline{cd}]}_1 X^B(y) f^A_{[\underline{cd}]B}$, and also

$$
\begin{aligned}
\{Q_R, J^{\alpha}_0(y)\} &= -\hat{\lambda}^{\hat{\alpha}} J^{\underline{c}}_0(y) f^{\alpha}_{\hat{\alpha}\underline{c}} \,, \\
\{Q_L, J^{\alpha}_0(y)\} &= \lambda^{\gamma} (\Phi^{[\underline{cd}]} + N^{[\underline{cd}]}_{\mu} \mathcal{P}^{\mu 0} + \hat{N}^{[\underline{cd}]}_{\mu} \tilde{\mathcal{P}}^{\mu 0})(y) f^{\alpha}_{\gamma[\underline{cd}]} - \nabla_1 \lambda^{\alpha}(y) \,, \\
\left\{Q_R, J^{\hat{\alpha}}_0(y)\right\} &= \hat{\lambda}^{\hat{\gamma}} (\Phi^{[\underline{cd}]} + N^{[\underline{cd}]}_{\mu} \mathcal{P}^{\mu 0} + \hat{N}^{[\underline{cd}]}_{\mu} \tilde{\mathcal{P}}^{\mu 0})(y) f^{\alpha}_{\hat{\gamma}[\underline{cd}]} + \nabla_1 \hat{\lambda}^{\hat{\alpha}}(y) \,, \\
\left\{Q_L, J^{\hat{\alpha}}_0(y)\right\} &= -\lambda^{\alpha} J^{\underline{c}}_0(y) f^{\hat{\alpha}}_{\alpha\underline{c}} \,.
\end{aligned} \tag{4.15}
$$

It turns out that we will also need the following Poisson brackets

$$
\begin{aligned}
\left\{Q_L, J^{[\underline{cd}]}_1(y)\right\} &= \lambda^{\alpha} J^{\hat{\gamma}}_1(y) f^{[\underline{cd}]}_{\alpha\hat{\gamma}} \,, \\
\left\{Q_R, J^{[\underline{cd}]}_1(y)\right\} &= \hat{\lambda}^{\hat{\alpha}} J^{\beta}_1(y) f^{[\underline{cd}]}_{\hat{\alpha}\beta} \,.
\end{aligned} \tag{4.16}
$$

The Poisson bracket between BRST charges and ghost fields can be easily worked out using (4.3) and we obtain

$$
\begin{aligned}
\{Q_{(L,R)}, \lambda^{\alpha}(y)\} &= \left\{Q_{(L,R)}, \hat{\lambda}^{\hat{\alpha}}(y)\right\} = 0 \,, \\
\{Q_L, \pi_{\alpha}(y)\} &= -K_{\alpha\hat{\beta}} [J^{\hat{\beta}}_0 - J^{\hat{\beta}}_1](y) \,, \qquad \{Q_R, \pi_{\alpha}(y)\} = 0 \,, \\
\{Q_R, \hat{\pi}_{\hat{\alpha}}(y)\} &= -K_{\hat{\alpha}\beta} [J^{\beta}_0 + J^{\beta}_1](y) \,, \qquad \{Q_L, \hat{\pi}_{\hat{\alpha}}(y)\} = 0 \,.
\end{aligned} \tag{4.17}
$$

In the same way we can determine the Poisson brackets between BRST charges and $N^{[\underline{cd}]}$, $\hat{N}^{[\underline{cd}]}$

$$
\begin{aligned}
\left\{Q_L, N^{[\underline{cd}]}(y)\right\} &= [J^{\hat{\beta}}_0 - J^{\hat{\beta}}_1] \lambda^{\alpha}(y) f^{[\underline{cd}]}_{\hat{\beta}\alpha} \,, \\
\left\{Q_R, \hat{N}^{[\underline{cd}]}(y)\right\} &= [J^{\beta}_0 + J^{\beta}_1] \hat{\lambda}^{\hat{\alpha}}(y) f^{[\underline{cd}]}_{\beta\hat{\alpha}} \,.
\end{aligned} \tag{4.18}
$$

Before we conclude this section we would like to briefly discuss the BRST transformations of the (light-cone) components of the currents

$$J_{\pm}^{A} = \frac{1}{\sqrt{2}}(J_0^A \pm J_1^A) \ . \tag{4.19}$$

It is rather straightforward to calculate the action of the BRST charges Q_R, Q_L on the (chiral) currents $J_{\pm}^{A}$. For illustration, let us consider the action of the charge $Q = Q_R + Q_L$ on the current $J_{\pm}^{\underline{c}}$. Using (4.14) we obtain

$$\begin{aligned}
\left\{Q, J_{+}^{\underline{c}}(y)\right\} &= -\hat{\lambda}^{\hat{\alpha}} J_{+}^{\hat{\beta}}(y) f_{\hat{\alpha}\hat{\beta}}^{\underline{c}} - \lambda^{\alpha} J_{+}^{\beta}(y) f_{\alpha\beta}^{\underline{c}} \,, \\
\left\{Q, J_{-}^{\underline{c}}(y)\right\} &= -\lambda^{\alpha} J_{-}^{\beta}(y) f_{\alpha\beta}^{\underline{c}} - \hat{\lambda}^{\hat{\alpha}} J_{-}^{\hat{\beta}}(y) f_{\hat{\alpha}\hat{\beta}}^{\underline{c}} \ .
\end{aligned} \tag{4.20}$$

In the same way we can calculate the action of the BRST charge Q on all remaining currents. Since the procedure is straightforward we will not report it here. However we have to stress one important point. It can be easily shown that the action of the BRST charges on the chiral currents that in the canonical formalism is defined as the Poisson bracket between BRST charge Q and corresponding current, does not fully coincide with the BRST transformation of currents given in [11]. This follows from the fact that our calculation is based on Hamiltonian formalism that is not manifestly covariant. Secondly, the transformation of the currents given in [11] is a combination of a BRST transformation and a gauge transformation. Unfortunately it is not completely clear to us how these transformations are related to the BRST transformations given here.

4.2 Hamiltonian

At this point we are ready to determine the Hamiltonian for the pure spinor string in $AdS_5 \times S_5$. Using the supergroup notation, we define the matter part of the Hamiltonian as

$$\begin{aligned}
H_{\text{matt}} = \int dx \mathrm{Str}(\partial_0 J_1 \Pi - \mathcal{L}_{\text{matt}}) = \int dx \Big(\frac{1}{2} \Big[J_0^{\underline{c}} J_0^{\underline{d}} K_{\underline{cd}} + J_1^{\underline{c}} J_1^{\underline{d}} K_{\underline{cd}} + J_0^{\alpha} J_0^{\hat{\beta}} K_{\alpha\hat{\beta}} \\
+ J_0^{\hat{\beta}} J_0^{\alpha} K_{\hat{\beta}\alpha} + J_1^{\alpha} J_1^{\hat{\beta}} K_{\alpha\hat{\beta}} + J_1^{\hat{\beta}} J_1^{\alpha} K_{\hat{\beta}\alpha} \Big] + N^{[\underline{cd}]} K_{[\underline{cd}][\underline{ef}]} J_1^{[\underline{ef}]} - \hat{N}^{[\underline{cd}]} K_{[\underline{cd}][\underline{ef}]} J_1^{[\underline{ef}]} \Big)
\end{aligned} \tag{4.21}$$

In the same way we define the ghost part of the Hamiltonian as

$$\begin{aligned}
H_{\text{ghost}} = & \int dx \left(\pi_{\alpha} \partial_0 \lambda^{\alpha} + \hat{\pi}_{\hat{\alpha}} \partial_0 \hat{\lambda}^{\hat{\alpha}} - \mathcal{L}_{\text{ghosts}} \right) = \\
& \int dx \left(-\pi_{\alpha} \partial_1 \lambda^{\alpha} + \hat{\pi}_{\hat{\alpha}} \partial_1 \hat{\lambda}^{\hat{\alpha}} + N^{[\underline{cd}]} K_{[\underline{cd}][\underline{ef}]} \hat{N}^{[\underline{ef}]} \right)
\end{aligned} \tag{4.22}$$

using the fact that

$$N_{\mu}^{[\underline{cd}]} \mathcal{P}^{\mu\nu} K_{[\underline{cd}][\underline{ef}]} \hat{N}_{\nu}^{[\underline{ef}]} = -N^{[\underline{cd}]} K_{[\underline{cd}][\underline{ef}]} \hat{N}^{[\underline{ef}]} \ . \tag{4.23}$$

Finally we introduce the Hamiltonian that corresponds to the SO(4,1) $\times$ SO(5) gauge symmetry constraints and to the pure spinor constraints (4.4)

$$H_{\text{cons}} = H_{\text{coset}} + H_{\text{pure}} \,,$$

JHEP08(2006)030

$$H_{\rm coset} = \int dx \Gamma_{[\underline{cd}]} \Phi^{[\underline{cd}]} \, ,$$
$$H_{\rm pure} = \int dx (\Gamma_{\underline{c}} \Phi^{\underline{c}} + \hat{\Gamma}_{\underline{c}} \hat{\Phi}^{\underline{c}}) \, , \tag{4.24}$$

where $\Gamma_{[\underline{cd}]} \, , \Gamma_{\underline{c}} \, , \hat{\Gamma}_{\underline{c}}$ are some *a priori* arbitrary functions of the phase space variables $(J_1^A, \Pi_A, \lambda, \hat{\lambda}, \pi, \hat{\pi})$.[4] Then the total Hamiltonian is equal to

$$H = H_{\rm matt} + H_{\rm ghost} + H_{\rm cons} \ . \tag{4.25}$$

The general theory of constrained systems requires that one make sure that the time evolution of the constraints does not generate any additional (secondary) ones [35]. Let us begin with $\Phi^{[\underline{cd}]}$ and prove that

$$\left\{ \Phi^{[\underline{cd}]}(x), H \right\} \approx 0 \ , \tag{4.26}$$

where $\approx$ means that this Poisson bracket vanishes on constraint surface $\Phi^{[\underline{cd}]} = 0$.

Firstly, it can be explicitly shown, using the Poisson brackets given in the previous section that

$$\left\{ \Phi^{[\underline{cd}]}(x), H_{\rm matt} + H_{\rm ghost} \right\} = 0 \ . \tag{4.27}$$

This result can be also considered as a consequence of the fact that $H_{\rm matt} + H_{\rm ghost}$ are manifestly gauge invariant. On the other hand the Poisson bracket of $\Phi^{[\underline{cd}]}$ with $H_{\rm coset}$ is equal to

$$\left\{ \Phi^{[\underline{cd}]}(x), H_{\rm coset} \right\} = \int dy \left\{ \Phi^{[\underline{cd}]}(x), \Gamma_{[\underline{ef}]}(y) \right\} \Phi^{[\underline{ef}]}(y) + \\ + \Gamma_{[\underline{ef}]} K^{[\underline{ef}][\underline{gh}]} f_{[\underline{gh}][\underline{ab}]}^{[\underline{cd}]} \Phi^{[\underline{ab}]}(x) \approx 0 \, , \tag{4.28}$$

where we have used (3.18). Finally, the Poisson bracket between $\Phi^{[\underline{cd}]}$ and $H_{\rm pure}$ can be easily calculated with the help of (4.9) and we get

$$\left\{ \Phi^{[\underline{cd}]}(x), H_{\rm pure} \right\} = \int dy (\left\{ \Phi^{[\underline{cd}]}(x), \Gamma_{\underline{e}}(y) \right\} \Phi^{\underline{e}}(y) + \left\{ \Phi^{[\underline{cd}]}(x), \hat{\Gamma}_{\underline{e}}(y) \right\} \hat{\Phi}^{\underline{e}}(y)) + \\ \Gamma_{\underline{e}} \Phi^{\underline{f}}(x) f_{\underline{f}[\underline{ab}]}^{\underline{e}} K^{[\underline{ab}][\underline{cd}]} + \hat{\Gamma}_{\underline{e}} \hat{\Phi}^{\underline{f}}(x) f_{\underline{f}[\underline{ab}]}^{\underline{e}} K^{[\underline{ab}][\underline{cd}]} \approx 0 \ . \tag{4.29}$$

In other words the Poisson bracket between $\Phi^{[\underline{cd}]}$ and H vanishes on constraint surface and hence the time evolution of $\Phi^{[\underline{cd}]}$ does not generate additional secondary constraint.

The situation is slightly more complicated in case of the pure spinor constraints (4.4). In fact, it is easy to see, using (4.6) that

$$\{ \Phi^{\underline{c}}(x), H_{\rm matt} \} \approx 0 \, , \quad \left\{ \hat{\Phi}^{\underline{c}}(x), H_{\rm matt} \right\} \approx 0 \ . \tag{4.30}$$

[4] It would be certainly interesting to perform a "more symmetric" analysis, whereby the generalized BRST operators include the constraints $\Phi^{[\underline{cd}]}, \Phi^{\underline{c}}, \hat{\Phi}^{\underline{c}}$, as suggested in [30]. We leave this analysis to future work.

JHEP08(2006)030

Moreover, we can also show in the same way as in (4.29) that the Poisson bracket between pure spinor constraints and H_{coset} vanishes on constraint surface. Finally, using (4.7) we can show that

$$\{\Phi^{\underline{c}}(x), H_{\text{pure}}\} = \left\{\hat{\Phi}^{\underline{c}}(x), H_{\text{pure}}\right\} = 0 \,. \tag{4.31}$$

On the other hand the Poisson brackets between $\Phi^{\underline{c}}$, $\hat{\Phi}^{\underline{c}}$ and H_{ghost} are equal to

$$\begin{aligned}
\{\Phi^{\underline{c}}(x), H_{\text{ghost}}\} &= -\partial_1\lambda^\alpha\gamma^{\underline{c}}_{\alpha\beta}\lambda^\beta(x) - \Phi^{\underline{d}} f^{\underline{c}}_{\underline{d}[\underline{ef}]}\hat{N}^{[\underline{ef}]}(x) = \\
&\qquad -\partial_1\Phi^{\underline{c}}(x) - \Phi^{\underline{d}} f^{\underline{c}}_{\underline{d}[\underline{ef}]}\hat{N}^{[\underline{ef}]}(x) \,, \\
\left\{\hat{\Phi}^{\underline{c}}(x), H_{\text{ghost}}\right\} &= \partial_1\hat{\lambda}^{\hat{\alpha}}\gamma^{\underline{c}}_{\hat{\alpha}\hat{\beta}}\hat{\lambda}^{\hat{\beta}}(x) - \hat{\Phi}^{\underline{d}} f^{\underline{c}}_{\underline{d}[\underline{ef}]}N^{[\underline{ef}]}(x) = \\
&\qquad \partial_1\hat{\Phi}^{\underline{c}}(x) - \hat{\Phi}^{\underline{d}} f^{\underline{c}}_{\underline{d}[\underline{ef}]}N^{[\underline{ef}]}(x) \,,
\end{aligned} \tag{4.32}$$

where we have used (4.5) and (4.6). We momentarily argue that these expressions vanish along the constraints $\Phi^{\underline{c}} = \hat{\Phi}^{\underline{c}} = 0$. It is obvious that this is true for the second terms on the second and the fourth line in (4.32). In order to clearly demonstrate that the first term on the second line in (4.32) vanishes along the constraints as well, note that it can be written as

$$\partial_1\Phi^{\underline{c}}(x) = \lim_{x'\to x}\frac{1}{(x'-x)}\left(\Phi^{\underline{c}}(x') - \Phi^{\underline{c}}(x)\right) \,. \tag{4.33}$$

In other words we can interpret this term as a difference of the constraints at different points $x = x'$. Since the constraint functions have to vanish for all x it is now clear that this difference vanishes as well. In the same way we can argue that the first term on the fourth line in (4.32) vanishes on the constraint surface $\hat{\Phi}^{\underline{c}} = 0$. In summary, the time evolution of the pure spinor constraints does not generate new secondary constraints.

5. Equations of motions

Using the form of the Hamiltonian (4.25) and the known Poisson brackets it is easy to determine the classical equations of motion for currents and ghosts. We explicitly determine these equations and show that they coincide with the equations of motion derived in the Lagrangian formalism [11, 22, 23], for an appropriate choice of the gauge parameters,

$$\tilde{\mathcal{P}}^{\mu\nu}\nabla_\mu J^{(3)}_\nu + [J^{(3)}_\nu, N_\mu]\mathcal{P}^{\mu\nu} + [J^{(3)}_\nu, \hat{N}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0 \,, \tag{5.1}$$

$$\mathcal{P}^{\mu\nu}\nabla_\mu J^{(1)}_\nu + [J^{(1)}_\nu, N_\mu]\mathcal{P}^{\mu\nu} + [J^{(1)}_\nu, \hat{N}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0 \,, \tag{5.2}$$

$$\mathcal{P}^{\mu\nu}\nabla_\mu J^{(2)}_\nu - \epsilon^{\mu\nu}[J^{(1)}_\mu, J^{(1)}_\nu] + [J^{(2)}_\nu, N_\mu]\mathcal{P}^{\mu\nu} + [J^{(2)}_\nu, \hat{N}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0 \,, \tag{5.3}$$

$$\tilde{\mathcal{P}}^{\mu\nu}\nabla_\mu J^{(2)}_\nu + \epsilon^{\mu\nu}[J^{(3)}_\mu, J^{(3)}_\nu] + [J^{(2)}_\nu, N_\mu]\mathcal{P}^{\mu\nu} + [J^{(2)}_\nu, \hat{N}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0 \,, \tag{5.4}$$

$$\mathcal{P}^{\mu\nu}\nabla_\nu\lambda - \mathcal{P}^{\mu\nu}[\lambda, \hat{N}_\nu] = 0 \,, \tag{5.5}$$

$$\tilde{\mathcal{P}}^{\mu\nu}\nabla_\nu\hat{\lambda} - \tilde{\mathcal{P}}^{\mu\nu}[\hat{\lambda}, N_\nu] = 0 \,, \tag{5.6}$$

where

$$\begin{gathered}
\nabla_\nu J^{(i)}_\mu = \partial_\nu J^{(i)}_\mu + [J^{(0)}_\nu, J^{(i)}_\mu] \,, \\
\nabla_\mu\lambda = \partial_\mu\lambda + [J^{(0)}_\mu, \lambda] \,, \quad \nabla_\mu\hat{\lambda} = \partial_\mu\hat{\lambda} + [J^{(0)}_\mu, \hat{\lambda}] \,,
\end{gathered} \tag{5.7}$$

and where we also used the notations defined in (2.3).

Let us now turn our attention onto the Hamiltonian formalism. Recall that the time dependence of any classical observable is governed by the equation

$$\partial_0 X = \{X, H\} \ . \tag{5.8}$$

We must also stress that we will write the resulting form of the equations of motion that is valid along the constraints $\Phi^{[\underline{cd}]} = \Phi^{\underline{c}} = \hat{\Phi}^{\underline{d}} = 0$.

Let us start with the equation of motion for $\lambda^\alpha, \hat{\lambda}^{\hat{\alpha}}$. Using

$$\left\{\lambda^\alpha(x), N^{[\underline{cd}]}(y)\right\} = K^{\alpha\hat{\beta}} f^{[\underline{cd}]}_{\hat{\beta}\gamma} \lambda^\gamma(x)\delta(x-y) \tag{5.9}$$

and

$$\left\{\hat{\lambda}^{\hat{\alpha}}(x), \hat{N}^{[\underline{cd}]}(y)\right\} = K^{\hat{\alpha}\beta} f^{[\underline{cd}]}_{\beta\hat{\gamma}} \hat{\lambda}^{\hat{\gamma}}(x)\delta(x-y) \tag{5.10}$$

and also using (4.25) we easily get the equation of motion for λ^α

$$\partial_0\lambda^\alpha = \{\lambda^\alpha, H\} = -\partial_1\lambda^\alpha - J_1^{[\underline{cd}]}\lambda^\gamma f^\alpha_{[\underline{cd}]\gamma} - \hat{N}^{[\underline{cd}]}\lambda^\gamma f^\alpha_{[\underline{cd}]\gamma} + \Gamma_{[\underline{cd}]}\lambda^\gamma f^{[\underline{cd}]}_{\gamma\hat{\beta}} K^{\hat{\beta}\alpha} \ . \tag{5.11}$$

As we know $\Gamma_{[\underline{cd}]}$ are arbitrary functions that reflect the gauge invariance of the theory. However we can fix the form of these parameters $\Gamma_{[\underline{cd}]}$ in order to obtain the form of the equation of motion that coincide with the covariant equation (5.5).Using the fact that

$$\Gamma_{[\underline{cd}]}\lambda^\gamma f^{[\underline{cd}]}_{\gamma\hat{\beta}} K^{\hat{\beta}\alpha} = \Gamma^{[\underline{cd}]}\lambda^\gamma f^\alpha_{[\underline{cd}]\gamma} \tag{5.12}$$

and by comparing (5.5) with (5.11) we see that it is natural to take

$$\Gamma^{[\underline{cd}]} = -J_0^{[\underline{cd}]} \ . \tag{5.13}$$

In what follows we will assume the choice (5.13) that in the end will lead to the equivalence of the equations of motion derived from the Hamiltonian formalism with the ones derived using the Lagrangian formalism.

The equation of motion for $\hat{\lambda}$ can be easily derived as in the case of the ghost λ and it coincides with (5.6) with the help of (5.13).

In the following, we will derive the equations of motion for matter variables $J^{(i)}$ using the Poisson brackets derived in the previous section and the matter Hamiltonian in (4.21).

Let us start with the equation of motion for $J_1^{\underline{c}}$

$$\begin{aligned} \partial_0 J_1^{\underline{c}} = \left\{J_1^{\underline{c}}, H\right\} = \\ \partial_1 J_0^{\underline{c}} + J_1^{[\underline{ef}]} J_0^{\underline{d}} f^{\underline{c}}_{[\underline{ef}]\underline{d}} + J_1^{\hat{\alpha}} J_0^{\hat{\beta}} f^{\underline{c}}_{\hat{\alpha}\hat{\beta}} + J_1^\alpha J_0^\beta f^{\underline{c}}_{\alpha\beta} - J_0^{[\underline{ef}]} J_1^{\underline{d}} f^{\underline{c}}_{[\underline{ef}]\underline{d}} \end{aligned} \tag{5.14}$$

that can be also written as

$$-\nabla_0 J_1^{(2)} + \nabla_1 J_0^{(2)} + [J_1^{(3)}, J_0^{(3)}] + [J_1^{(1)}, J_0^{(1)}] = 0 \ . \tag{5.15}$$

On the other hand the equation of motion for $J_0^{\underline{c}}$ is more involved and takes the form

$$\begin{aligned} \partial_0 J_0^{\underline{c}} = \left\{J_0^{\underline{c}}, H\right\} = J_\nu^{\underline{d}} N_\mu^{[\underline{ef}]} f^{\underline{c}}_{\underline{d}[\underline{ef}]}\mathcal{P}^{\mu\nu} + J_\nu^{\underline{d}} \hat{N}_\mu^{[\underline{ef}]} f^{\underline{c}}_{\underline{d}[\underline{ef}]}\tilde{\mathcal{P}}^{\mu\nu} + \\ +\partial_1 J_1^{\underline{c}} + J_1^{[\underline{ef}]} J_1^{\underline{d}} f^{\underline{c}}_{[\underline{ef}]\underline{d}} - J_0^{[\underline{ef}]} J_0^{\underline{d}} f^{\underline{c}}_{[\underline{ef}]\underline{d}} \ , \end{aligned} \tag{5.16}$$

or equivalently

$$\nabla_1 J_1^{(2)} - \nabla_0 J_0^{(2)} + [J_\nu^{(2)}, N_\mu]\mathcal{P}^{\mu\nu} + [J_\nu^{(2)}, \hat{N}_\mu]\tilde{\mathcal{P}}^{\mu\nu} - [J_1^{(3)}, J_0^{(3)}] + [J_1^{(1)}, J_0^{(1)}] = 0 \ . \quad (5.17)$$

On the one hand, if we sum (5.15) with (5.17) we obtain the equation that coincides with (5.3). On the other hand, if we take the difference of equations (5.15) and (5.17) we get an equation that coincides with (5.4).

Let us now consider the equation of motion for J_1^α. After some manipulations it can be written as

$$\begin{aligned} &- \nabla_0 J_0^{(1)} + \nabla_1 J_1^{(1)} + [J_0^{(3)}, J_1^{(2)}] - [J_1^{(3)}, J_0^{(2)}] + \\ &\qquad + [J_\nu^{(1)}, N_\mu]\mathcal{P}^{\mu\nu} + [J_\nu^{(1)}, \hat{N}_\mu]\tilde{\mathcal{P}}^{\mu\nu} = 0 \ . \end{aligned} \quad (5.18)$$

In the same way we can proceed with the equations of motion for J_0^α that in the compact notation takes the form

$$\nabla_1 J_0^{(1)} - \nabla_0 J_1^{(1)} + [J_1^{(3)}, J_0^{(2)}] - [J_0^{(3)}, J_1^{(2)}] = 0 \ . \quad (5.19)$$

It is easy to see that the if we add together (5.18) with (5.19) we derive the equation (5.2). In the same way we can show that the equations of motion for $J_1^{\hat{\alpha}}, J_0^{\hat{\alpha}}$ derived in the Hamiltonian formalism imply the equation (5.1).

6. Conservation and nihilpotency of the BRST charges

In this section we will show that the commutator of the BRST charges Q_R, Q_L with the Hamiltonian vanishes provided the dynamics is restricted to satisfy the local $SO(4,1) \times SO(5)$ constraint $\Phi^{[\underline{cb}]} = 0$ and the pure spinor constraint for the ghost fields. As a first step, we determine the Poisson bracket between H_{matt} and Q_L. Using the Poisson brackets given in (4.14), (4.15) and (4.16) we obtain

$$\begin{aligned} \{Q_L, H_{\text{matt}}\} = \int dx (& -\partial_1 \lambda^\alpha K_{\alpha\hat{\beta}}(J_0^{\hat{\beta}} - J_1^{\hat{\beta}}) + (J_0^{\hat{\beta}} - J_1^{\hat{\beta}})\lambda^\gamma f_{\hat{\beta}\gamma}^{[\underline{cd}]} K_{[\underline{cd}][\underline{ef}]} \hat{N}^{[\underline{ef}]} \\ & + \lambda^\gamma \Phi^{[\underline{cd}]} f_{\gamma[\underline{cd}]}^\alpha K_{\alpha\hat{\beta}} J_0^{\hat{\beta}} + \lambda^\gamma N^{[\underline{cd}]} f_{[\underline{cd}]\gamma}^\alpha K_{\alpha\hat{\beta}}(J_0^{\hat{\beta}} + J_1^{\hat{\beta}})) \ . \end{aligned} \quad (6.1)$$

On the other hand the Poisson bracket of Q_L with H_{ghost} can be easily worked out using (4.17) and (4.18) and we get

$$\{Q_L, H_{\text{ghost}}\} = \int dx (\partial_1 \lambda^\alpha K_{\alpha\hat{\beta}}[J_0^{\hat{\beta}} - J_1^{\hat{\beta}}] - [J_0^{\hat{\beta}} - J_1^{\hat{\beta}}]\lambda^\alpha f_{\hat{\beta}\alpha}^{[\underline{cd}]} K_{[\underline{cd}][\underline{ef}]} \hat{N}^{[\underline{ef}]}) \ . \quad (6.2)$$

Finally the Poisson bracket of Q_L with H_{coset} is equal to

$$\{Q_L, H_{\text{coset}}\} = \int dx \left\{Q_L, \Gamma_{[\underline{cd}]}(x)\right\} \Phi^{[\underline{cd}]}(x) \, , \quad (6.3)$$

where we have used the fact that $\left\{Q_L, \Phi^{[\underline{cd}]}(x)\right\} = 0$. In the same way we can show that

$$\{Q_L, H_{\text{pure}}\} \approx 0 \ . \quad (6.4)$$

JHEP08(2006)030

Collecting all these results we obtain that the commutator of Q_L with H is equal to

$$\begin{aligned}\{Q_L, H\} &= \int dx(\lambda^\gamma f^\alpha_{\gamma\underline{[cd]}} K_{\alpha\hat{\beta}} J_0^{\hat{\beta}} + \{Q_L, \Gamma_{\underline{[cd]}}(x)\})\Phi^{\underline{[cd]}}(x) \\ &+ \int dx \lambda^\gamma N^{\underline{[cd]}} f^\alpha_{\underline{[cd]}\gamma} K_{\alpha\hat{\beta}}(J_0^{\hat{\beta}} + J_1^{\hat{\beta}}) \ . \end{aligned} \tag{6.5}$$

The expression on the first line in (6.5) is proportional to $\Phi^{\underline{[cd]}}$ that is zero on the constraint surface. On the other hand using the explicit form of $N^{\underline{[cd]}}$ we can rewrite the expression on the second line in (6.5), omitting the factor $K_{\alpha\hat{\beta}}(J_0 - J_1)^{\hat{\beta}}$, as

$$\lambda^\gamma N^{\underline{[cd]}} f^\alpha_{\underline{[cd]}\gamma} = \pi_\beta K^{\beta\hat{\beta}} f^{\underline{[cd]}}_{\hat{\beta}\delta} \lambda^\delta f^\alpha_{\underline{[cd]}\gamma} \lambda^\gamma = \frac{1}{2} \pi_\beta K^{\beta\hat{\beta}} f^\alpha_{\hat{\beta}\underline{c}} (\lambda^\delta f^{\underline{c}}_{\delta\gamma} \lambda^\gamma) \ , \tag{6.6}$$

where in the final step we have used the generalized Jacobi identity (3.15). However since $f^{\underline{c}}_{\delta\gamma} = 2(\gamma^{\underline{c}})_{\delta\gamma}$ we obtain that the BRST charge Q_L is conserved on the constraint surface

$$\Phi^{\underline{[cd]}} = \Phi^{\underline{c}} = 0 \ . \tag{6.7}$$

In the same way we can calculate the Poisson bracket of Q_R with H and we obtain

$$\begin{aligned}\{Q_R, H\} &= \int dx(-J_0^\alpha \hat{\lambda}^{\hat{\gamma}}(x) K_{\alpha\hat{\alpha}} f^{\hat{\alpha}}_{\hat{\gamma}\underline{[cd]}} + \{Q_R, \Gamma_{\underline{[cd]}}(x)\})\Phi^{\underline{[cd]}}(x) \\ &- \int dx \hat{N}^{\underline{[cd]}} f^{\hat{\alpha}}_{\underline{[cd]}\hat{\gamma}} \hat{\lambda}^{\hat{\gamma}} K_{\hat{\alpha}\alpha}(J_0^\alpha + J_1^\alpha) \ . \end{aligned} \tag{6.8}$$

The expression on the first line in (6.8) is again proportional to the constraint $\Phi^{\underline{[cd]}}$ and hence it vanishes on the constraint surface $\Phi^{\underline{[cd]}} = 0$. On the other hand the expression on the second line is proportional to

$$\begin{aligned}-\hat{N}^{\underline{[cd]}} f^{\hat{\alpha}}_{\underline{[cd]}\hat{\gamma}} \hat{\lambda}^{\hat{\gamma}} K_{\hat{\alpha}\alpha} &= -\pi_\gamma K^{\gamma\hat{\delta}} f^{\underline{[cd]}}_{\hat{\delta}\hat{\beta}} \hat{\lambda}^{\hat{\beta}} f^{\hat{\alpha}}_{\underline{[cd]}\hat{\gamma}} \hat{\lambda}^{\hat{\gamma}} = \\ &= \frac{1}{2} \pi_\gamma K^{\gamma\hat{\delta}} f^{\hat{\alpha}}_{\hat{\delta}\underline{c}} (\hat{\lambda}^{\hat{\gamma}} f^{\underline{c}}_{\hat{\gamma}\hat{\beta}} \hat{\lambda}^{\hat{\beta}}) \sim \hat{\Phi}^{\underline{c}} \end{aligned} \tag{6.9}$$

and we see that the Hamiltonian 'commutes' or, rather, is in involution with Q_R along the constraints. In other words we have shown that the BRST charges are conserved as expected for any generator of a global symmetries.

It is also important to prove that the BRST charges are nihilpotent at least on the constraint surfaces $\Phi^{\underline{[cd]}} = \Phi^{\underline{c}} = \hat{\Phi}^{\underline{c}} = 0$. In other words we have to show that the Poisson brackets between Q_R, Q_L vanish or they are proportional to generators of gauge transformations. In fact, using the known form of the Poisson bracket between BRST generator Q_L and the currents J^A we easily obtain

$$\begin{aligned}\{Q_L, Q_L\} &= -\int dx \lambda^\alpha K_{\alpha\hat{\beta}} \left(\left\{ Q_L, J_0^{\hat{\beta}}(x) \right\} - \left\{ Q_L, J_1^{\hat{\beta}}(x) \right\} \right) = \\ &= \int dx \lambda^\alpha K_{\alpha\hat{\beta}} f^{\hat{\beta}}_{\gamma\underline{c}} \lambda^\gamma [J_0^{\underline{c}} - J_1^{\underline{c}}] \ . \end{aligned} \tag{6.10}$$

Using the fact that

$$\lambda^\alpha K_{\alpha\hat\beta} f^{\hat\beta}_{\gamma\underline{c}} \lambda^\gamma = \lambda^\gamma f^{\underline{d}}_{\gamma\alpha} \lambda^\alpha K_{\underline{dc}} \sim \Phi^{\underline{d}} \tag{6.11}$$

we obtain the result that the BRST charge Q_L is nihilpotent as a consequence of the pure spinor constraint (4.4). We would like to stress that our proof that Q_L is nihilpotent is valid even if all fields are off-shell. It only relies on the local $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ and pure spinor constraints.

In case of Q_R we proceed in the same way and we find that Q_R is nihilpotent as well.

Finally, we can calculate the Poisson bracket between Q_R and Q_L

$$\begin{aligned} \{Q_L, Q_R\} &= -\int dx \hat\lambda^{\hat\alpha} K_{\hat\alpha\alpha} [\{Q_L, J_0^\alpha(x)\} + \{Q_L, J_1^\alpha(x)\}] = \\ &= -\int dx \hat\lambda^{\hat\alpha} K_{\hat\alpha\alpha} \lambda^\gamma (\Phi^{[\underline{cd}]} + N^{[\underline{cd}]} + \hat N^{[\underline{cd}]}) f^\alpha_{\gamma[\underline{cd}]} \,. \end{aligned} \tag{6.12}$$

It is convenient to rewrite the term proportional to $N^{[\underline{cd}]}$ as

$$-\lambda^\gamma N^{[\underline{cd}]} f^\alpha_{\gamma[\underline{cd}]} = -\frac{1}{2} \lambda^\gamma f^{\underline{c}}_{\gamma\delta} \lambda^\delta f^\alpha_{\underline{c}\hat\beta} K^{\hat\beta\beta} \pi_\beta \sim \Phi^{\underline{c}} \tag{6.13}$$

and we see that it vanishes on the constraint surface $\Phi^{\underline{c}} = 0$. In the same way we can show

$$-\hat\lambda^{\hat\alpha} K_{\hat\alpha\alpha} \hat N^{[\underline{cd}]} f^\alpha_{\gamma[\underline{cd}]} = -\frac{1}{2} \hat\lambda^{\hat\alpha} f^{\underline{c}}_{\hat\alpha\hat\delta} \hat\lambda^{\hat\delta} f^{\hat\beta}_{\beta\underline{c}} \hat\pi_{\hat\gamma} K^{\hat\gamma\beta} K_{\hat\beta\alpha} \sim \hat\Phi^{\underline{c}} \tag{6.14}$$

that vanishes on the constraint surface $\hat\Phi^{\underline{c}} = 0$. Finally, the first term in (6.12) is proportional to $\Phi^{[\underline{cd}]}$ and hence it vanishes on the constraint surface $\Phi^{[\underline{cd}]} = 0$.

Let us summarize the results presented in this section. We have shown that the Poisson brackets between BRST generators vanish on the constraint surface. It is important that this result holds without assuming that the fundamental fields obey the equations of motion. We also hope that this result can be considered as an additional support to the analysis performed in [12]. It would be certainly very interesting to extend this analysis to the full quantum theory and further explore the consequence of the non-chiral splitting of the currents.

7. Global currents and integrability

We would now like to study the classically conserved local currents, that generate global $PSU(2,2|4)$ transformations, and their non-local extensions, whose conservation strongly supports classical integrability of the theory [39, 37, 23, 40, 34, 41], within the present approach.

In the covariant pure spinor formalism the problem has been studied by Vallilo [23]. One starts with a new set of left-invariant currents $\hat J(u)$ satisfying the flatness condition

$$d\hat J + \hat J \wedge \hat J = 0 \tag{7.1}$$

for any value of the spectral parameter u with the 'initial' condition $\hat{J}(0) = J = g^{-1}dg$. Making an ansatz of the form

$$\begin{aligned}\hat{J}_\mu(u) = J_\mu &+ \frac{1}{2}\mathcal{P}_{\mu\nu}[a(u)J^{\nu(2)} + b(u)J^{\nu(1)} + c(u)J^{\nu(3)} + \tilde{d}(u)\hat{N}^\nu] \\ &+ \frac{1}{2}\tilde{\mathcal{P}}_{\mu\nu}[\tilde{a}(u)J^{\nu(2)} + \tilde{b}(u)J^{\nu(1)} + \tilde{c}(u)J^{\nu(3)} + d(u)N^\nu]\end{aligned} \tag{7.2}$$

and imposing flatness, using flatness of $J_\mu(0)$ and the classical field equations, derived above in a Hamiltonian form or in [23, 12, 22] in a Lagrangian form, one gets[5]

$$\begin{aligned} a &= e^u - 1 \qquad \tilde{a} = e^{-u} - 1 \\ b &= e^{3u/2} - 1 \qquad \tilde{b} = e^{-u/2} - 1 \\ c &= e^{u/2} - 1 \qquad \tilde{c} = e^{-3u/2} - 1 \\ d &= e^{2u} - 1 \qquad \tilde{d} = e^{-2u} - 1 \end{aligned} \tag{7.3}$$

so that eventually

$$\begin{aligned}\hat{J}_\mu(u) = J_\mu + (\eta_{\mu\nu}(\cosh u - 1) + \epsilon_{\mu\nu}\sinh u)J^{\nu(2)} +\\ (\eta_{\mu\nu}(\cosh u e^{u/2} - 1) + \epsilon_{\mu\nu}\sinh u e^{u/2})J^{\nu(1)} +\\ (\eta_{\mu\nu}(\cosh u e^{-u/2} - 1) + \epsilon_{\mu\nu}\sinh u e^{-u/2})J^{\nu(3)} +\\ +\sinh u e^u \tilde{\mathcal{P}}_{\mu\nu}N^\nu - \sinh u e^{-u}\mathcal{P}_{\mu\nu}\hat{N}^\nu \ .\end{aligned} \tag{7.4}$$

Flatness of the current $\hat{J}$ implies integrability for any u of the equation

$$\hat{D}_\mu \chi = 0 \tag{7.5}$$

where $\hat{D}_\mu = \partial_\mu + \hat{J}_\mu$. It turns out to be convenient to exploit the combination

$$\epsilon_{\mu\nu}\partial^\nu\chi = -\epsilon_{\mu\nu}\hat{J}^\nu\chi + u\partial_\mu\chi + u\hat{J}_\mu\chi \quad . \tag{7.6}$$

Setting $\hat{A}_\mu(u) = \hat{J}_\mu(u) - J_\mu = ug^{-1}a_\mu(u)g$ (since $\hat{A}_\mu(0) = 0$) one gets

$$\epsilon_{\mu\nu}\partial^\nu(g\chi) = u\partial_\mu(g\chi) + (u^2 a_\mu(u) - u\epsilon_{\mu\nu}a^\nu(u))(g\chi) \quad . \tag{7.7}$$

Expanding χ and a_μ in powers of u around $u = 0$, one gets

$$\epsilon_{\mu\nu}\partial^\nu(g\chi_n) = \partial_\mu(g\chi_{n-1}) - \sum_{k=0}^{n-1}[\epsilon_{\mu\nu}a_k^\nu - a_{\mu,k-1}](g\chi_{n-k-1}) \tag{7.8}$$

The lowest order yields

$$n = 0 \qquad \partial^\nu(g\chi_0) = 0 \tag{7.9}$$

[5]Our spectral parameter u is related to the spectral parameter μ of [23] by $\mu = e^u$. Note also that we have chosen one particular solution from the ones found in [23] in order to obey the initial condition $\hat{J}_\mu(0) = J_\mu$. It is remarkable that the classical theory admits the same two one-parameter families of flat currents if one sets the contribution of the pure spinor ghost N to zero.

that implies $\chi_0 = Cg^{-1}$, where C is a constant that we can set to $C = 1$ henceforth for simplicity. Plugging the latter in the second equation yields

$$n = 1 \quad \epsilon_{\mu\nu}\partial^\nu(g\chi_1) = -\epsilon_{\mu\nu}a_0^\nu \tag{7.10}$$

which in turn implies that $j_{\mu,0} = \epsilon_{\mu\nu}a_0^\nu$ is a classically conserved local current. In the pure spinor approach one finds

$$j_{\mu,0} = g\left[J_\mu^{(2)} + J_\mu^{(1)} + J_\mu^{(3)} + \frac{1}{2}\epsilon_{\mu\nu}(J^{\nu(1)} - J^{\nu(3)}) + \tilde{\mathcal{P}}_{\mu\nu}N^\nu + \mathcal{P}_{\mu\nu}\hat{N}^\nu)\right]g^{-1} \tag{7.11}$$

Notice the difference w.r.t. the GS approach where

$$j_{\mu,0}^{\mathrm{GS}} = g\left[J_\mu^{(2)} + \frac{1}{2}\epsilon_{\mu\nu}(J_\mu^{\nu(1)} - J^{\nu(3)})\right]g^{-1} \tag{7.12}$$

in addition to the pure spinor contribution, absent in the GS approach, there is also an extra contribution in $J_\mu^{(1)}$ and $J_\mu^{(3)}$ since they appear in the kinetic term and not only in the WZ term, as required by κ symmetry which is instead fixed in the pure spinor approach.

The components $j_{\mu,0}^A = Str(T^A j_{\mu,0})$ of the conserved currents are expected to satisfy classical graded Poisson brackets encoding the structure of the global $PSU(2,2|4)$ algebra.

As anticipated the procedure can be pushed forward to identify the non-local currents. The first one arises at the next order where one finds

$$j_{1\mu} = \epsilon_{\mu\nu}\partial^\nu(g\chi_2) = -\epsilon_{\mu\nu}(a_0^\nu g\chi_1 + a_1^\nu) \tag{7.13}$$

where

$$a_{\mu,1} = g[J_\mu^{(2)} + \frac{5}{8}(J_\mu^{(1)} + J_\mu^{(3)}) + \frac{1}{2}\epsilon_{\mu\nu}(J^{\nu(1)} - J^{\nu(3)}) + \tilde{\mathcal{P}}_{\mu\nu}N^\nu + \mathcal{P}_{\mu\nu}\hat{N}^\nu]g^{-1} \tag{7.14}$$

and

$$g\chi_1 = -\frac{1}{\partial^2}(\partial_\mu a_0^\mu) = \frac{1}{\partial^2}(\epsilon^{\mu\nu}\partial_\mu j_{0,\nu}) \tag{7.15}$$

so that

$$j_{1\mu} = j_{1\mu}\frac{1}{\partial^2}(\epsilon^{\lambda\nu}\partial_\lambda j_{0,\nu}) - \epsilon_{\mu\nu}a_1^\nu \tag{7.16}$$

and so on.

The classically conserved non local currents generate a Yangian that has been studied for instance in [37, 38].

8. Conclusions

The present investigation has been devoted to a classical Hamiltonian analysis of the type IIB superstring on $AdS_5 \times S^5$ in the pure spinor approach. Following [32, 33], we have taken the spatial components of the (super)currents as canonical variables. In particular, we have computed the classical graded Poisson brackets of the left-invariant (super)currents and identified the first class constraints associated to the gauging of $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$. We have then studied the properties of the BRST generators and the Hamiltonian that governs the dynamics of the system compatibly with the local $SO(4,1) \times SO(5)$ and pure spinor constraints. Contrary to the standard GS approach, whereby fermionic constraints are both first and second class, the former being associated to local κ symmetry, the latter to the Dirac constraint, all the constraints we have found are first class and can be interpreted as generators of local symmetries. They appear in the classical Hamiltonian via suitable Lagrange multipliers. For a natural choice of the latter, we have satisfactorily shown equivalence of the canonical equations of motion with the covariant ones. Finally we have briefly discussed the global symmetries and the issue of integrability within the present framework.

It would be very interesting to further study the structure of the classical global algebra, that includes the global $PSU(2,2|4)$ symmetry, and its representations. A crucial step towards understanding the structure and classifying the classical string configurations ("motions") is determining the action of the currents on the fundamental fields, either the coset representative g or the spatial components of the left-invariant (super)currents, and ghosts. The latter are inescapably tangled with the 'matter' fields due to their non trivial transformations under space-time symmetries. One could then tackle the much harder issue of quantizing the string in this background and, in particular, finding the spectrum of excitations beyond the "massless" supergravity states.

Acknowledgments

We would like to thank A. Das, P. A. Grassi, H. Samtleben, Ya. Stanev for useful discussions. During completion of this work M.B. was visiting the Galileo Galilei Institute for Theoretical Physics of Arcetri (FI), INFN is acknowledged for hospitality and support. M.B. would also like to thank the organizers (C. Angelatonj, E. Dudas, T. Gherghetta and A. Pomarol) and the participants to the workshop "Beyond the Standard Model" for creating a stimulating environment. This work was supported in part by INFN, by the MIUR-COFIN contract 2003-023852, by the EU contracts MRTN-CT-2004-503369 and MRTN-CT-2004-512194, by the INTAS contract 03-516346 and by the NATO grant PST.CLG.978785. The work of J.K. is also supported in part by the Czech Ministry of Education under Contract No. MSM 0021622409.

A. Properties of $PSU(2,2|4)$

In this appendix, we briefly review the properties of the superalgebra $psu(2,2|4)$, for more details we recommend the papers [11, 12, 31, 34].

The generators of $psu(2,2|4)$ satisfy the graded commutation relations

$$T_A T_B - (-1)^{|A||B|} T_B T_A = f^C_{\mathrm{AB}} T_C \,, \tag{A.1}$$

The (super)index $A = (c, [cd], c', [c'd'], \alpha, \hat{\alpha})$ runs over the tangent space indices of the super-Lie algebra of $PSU(2,2|4)$, so that $(c, [cd])$ with $c, d = 0, \ldots, 4$ describe the $SO(4,2)$ isometries of AdS_5 and $(c', [c'd'])$ with $c', d' = 5 \ldots, 9$ describe the $SO(6)$ isometries of S^5. We also preserve the notation α and $\hat{\alpha}$ for the two 16-component Majorana-Weyl spinors.

Finally, $\underline{c}$ stands either for c or c' (10 (pseudo)translations). In the same way $[\underline{cd}]$ stands either for $[cd]$ or for $[c'd']$ (10+10 (pseudo)rotations').

The non-vanishing structure constants $f_{\rm AB}^{C}$ are

$$
\begin{aligned}
&f_{\alpha\beta}^{\underline{c}} = 2\gamma_{\alpha\beta}^{\underline{c}}\,, \quad f_{\hat\alpha\hat\beta}^{\underline{c}} = 2\gamma_{\hat\alpha\hat\beta}^{\underline{c}}\,,\\
&f_{\alpha\hat\beta}^{[\underline{ef}]} = f_{\hat\beta\alpha}^{[\underline{ef}]} = (\gamma^{\rm ef})_{\alpha}^{\gamma}\delta_{\gamma\hat\beta}\,, \quad f_{\alpha\hat\beta}^{[e'f']} = f_{\hat\beta\alpha}^{[e'f']} = -(\gamma^{e'f'})_{\alpha}^{\gamma}\delta_{\gamma\hat\beta}\,,\\
&f_{\alpha\underline{c}}^{\hat\beta} = -f_{\underline{c}\alpha}^{\hat\beta} = \frac{1}{2}(\gamma_{\underline{c}})_{\alpha\beta}\delta^{\beta\hat\beta}\,, \quad f_{\hat\alpha\underline{c}}^{\beta} = -f_{\underline{c}\hat\alpha}^{\beta} = -\frac{1}{2}(\gamma_{\underline{c}})_{\hat\alpha\hat\beta}\delta^{\beta\hat\beta}\,,\\
&f_{\rm cd}^{[ef]} = \frac{1}{2}\delta_{c}^{[e}\delta_{d}^{f]}\,, \quad f_{c'd'}^{[e'd']} = -\frac{1}{2}\delta_{c'}^{[e'}\delta_{d'}^{f']}\,, \quad f_{[\underline{cd}]\underline{e}}^{\underline{f}} = -f_{\underline{e}[\underline{cd}]}^{\underline{f}} = \eta_{\underline{e}[\underline{c}}\delta_{\underline{d}]}^{\underline{f}}\,,\\
&f_{[\underline{cd}][\underline{ef}]}^{[\underline{gh}]} = \frac{1}{2}\left(\eta_{\underline{ce}}\delta_{\underline{d}}^{[\underline{g}}\delta_{\underline{f}}^{\underline{h}]} - \eta_{\underline{cf}}\delta_{\underline{d}}^{[\underline{g}}\delta_{\underline{c}}^{\underline{h}]} + \eta_{\underline{df}}\delta_{\underline{c}}^{[\underline{g}}\delta_{\underline{e}}^{\underline{h}]} - \eta_{\underline{de}}\delta_{\underline{c}}^{[\underline{g}}\delta_{\underline{f}}^{\underline{h}]}\right)\,,\\
&f_{[\underline{cd}]\alpha}^{\beta} = -f_{\alpha[\underline{cd}]}^{\beta} = \frac{1}{2}(\gamma_{\underline{cd}})_{\alpha}^{\beta}\,, \quad f_{[\underline{cd}]\hat\alpha}^{\hat\beta} = -f_{\hat\alpha[\underline{cd}]}^{\hat\beta} = \frac{1}{2}(\gamma_{\underline{cd}})_{\hat\alpha}^{\hat\beta}\,.
\end{aligned}
\tag{A.2}
$$

The graded-symmetric Cartan-Killing supermetric

$$K_{\rm AB} = {\rm Str}(T_A T_B) = (-1)^{|A||B|}K_{\rm BA}\,, \tag{A.3}$$

with $|A| = 1$ if A is associated to a Grassmann odd generator and $|A| = 0$ if it is Grassmann even.

An essential feature of the superalgebra $psu(2,2|4)$ is that it admits $\mathbf{Z}_4$ automorphism Ω such that the condition $\Omega(\mathcal{H}) = \mathcal{H}$ determines the maximal subgroup ${\rm SO}(4,1)\times{\rm SO}(5)$ that has to be quotiented in the definition of the coset.

The $\mathbf{Z}_4$ authomorphism Ω allows us to decompose the superalgebra $\mathcal{G}$ as

$$\mathcal{G} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3\,, \tag{A.4}$$

where $\mathcal{H}_p$ denotes the eigenspace of Ω such that if $\mathbf{h}_p \in \mathcal{H}_p$ then

$$\Omega(\mathbf{h}_p) = i^p \mathbf{h}_p\,. \tag{A.5}$$

As we have argued above $\Omega(\mathbf{h}_0) = \mathbf{h}_0$ determines $\mathcal{H}_0 = {\rm SO}(4,1)\times{\rm SO}(5)$. $\mathcal{H}_2$ includes the remaining bosonic generators of the superalgebra, while $\mathcal{H}_1, \mathcal{H}_3$ consist of the fermionic generators of the algebra. The authomorphism Ω also implies a $\mathbf{Z}_4$ grading of the (anti)commutation relations

$$[\mathcal{H}_p, \mathcal{H}_q] \in \mathcal{H}_{p+q \ ({\rm mod}\ 4)}\,. \tag{A.6}$$

The generators of subspaces $\mathcal{H}^{(i)}$ are denoted as

$$\mathcal{H}_0:\ T_{[\underline{cd}]}\,, \qquad \mathcal{H}_1:\ T_\alpha\,, \qquad \mathcal{H}_2:\ T_{\underline{c}}\,, \qquad \mathcal{H}_3: T_{\hat\alpha}\,. \tag{A.7}$$

Then we can write the current J_μ as

$$J_\mu = J_\mu^A T_A = J_\mu^{(0)} + J_\mu^{(1)} + J_\mu^{(2)} + J_\mu^{(3)}\,, \tag{A.8}$$

$$J_\mu^{(0)} = J_\mu^{[\underline{cd}]}T_{[\underline{cd}]}\,, \qquad J_\mu^{(1)} = J_\mu^{\alpha}T_\alpha\,, \qquad J_\mu^{(2)} = J_\mu^{\underline{c}}T_{\underline{c}}\,, \qquad J_\mu^{(3)} = J_\mu^{\hat\alpha}T_{\hat\alpha}\,, \tag{A.9}$$

JHEP08(2006)030

where $A = (a, \alpha, \underline{c}, [\underline{cd}])$ and where J_μ^α, $J_\mu^{\hat{\alpha}}$ are Grassmann odd vectors, while $J_\mu^{[\underline{cd}]}$, $J_\mu^{\underline{c}}$ are Grassmann even vectors. The Killing form $\langle \mathcal{H}_p, \mathcal{H}_q \rangle$, defined in terms of a supertrace[6], is also $\mathbf{Z}_4$ invariant and hence we have

$$\langle \mathcal{H}_p, \mathcal{H}_q \rangle = 0 \,, \text{unless } p + q = 0 \text{ mod } 4 \,. \tag{A.10}$$

Using the relation (A.10) we find that the the Cartan-Killing (super)metric (A.3) takes the form

$$K_{\mathrm{AB}} = \begin{pmatrix} \kappa_{[\underline{cd}][\underline{ef}]} & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa_{\alpha\hat{\beta}} \\ 0 & 0 & \eta_{\underline{cd}} & 0 \\ 0 & \kappa_{\hat{\alpha}\beta} & 0 & 0 \end{pmatrix} \,. \tag{A.11}$$

Finally we also note that the structure constant of the $psu(2,2|4)$ algebra obey the graded (anti) symmetry property

$$f_{\mathrm{AB}}^{D} K_{\mathrm{DC}} = -(-1)^{|A||B|} f_{\mathrm{BA}}^{D} K_{\mathrm{DC}} = -(-1)^{|B||C|} f_{\mathrm{AC}}^{D} K_{\mathrm{DB}} \,. \tag{A.12}$$

B. Illustration of the Hamiltonian procedure

In this appendix we will demonstrate that the canonical approach given in section 3 can be easily applied to the case of a free massless boson.[7] Let us start with the action

$$S = -\frac{1}{2} \int d^2x \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \,. \tag{B.1}$$

In the standard Hamiltonian treatment we consider ϕ as canonical variable with the conjugate momentum $P = \partial_0 \phi$ and with the standard Poisson brackets

$$\{\phi(x), P(y)\} = \delta(x - y) \,. \tag{B.2}$$

On the other hand let us introduce the group element $g = e^\phi$. Then

$$j_\mu = g^{-1} \partial_\mu g = \partial_\mu \phi \tag{B.3}$$

and hence the action (B.1) can be written as

$$S = \frac{1}{2} \int d^2x (j_0 j_0 - j_1 j_1) \tag{B.4}$$

It is obvious that the current j_μ obeys the flatness condition

$$\partial_\mu j_\nu - \partial_\mu j_\nu = 0 \tag{B.5}$$

that allows one to express j_0 as

$$j_0 = \frac{1}{\partial_1} \partial_0 j_1 \tag{B.6}$$

[6]We define the supertrace Str in such a way that $\mathrm{Str}(M) = \mathrm{Tr}A - \mathrm{Tr}B$ if M is an even supermatrix and $\mathrm{Str}(M) = \mathrm{Tr}A + \mathrm{Tr}B$ if M is an odd supermatrix.

[7]We thank H. Samtleben for e-mail exchange on this.

and hence we can interpret j_1 as canonical variable. If we define the conjugate momentum as $\pi = \delta S / \delta \partial_0 j$ and use (B.6) we obtain

$$\pi = -\frac{1}{\partial_1^2}(\partial_0 j_1)\,, \tag{B.7}$$

where it is understood that j_1 obeys appropriate boundary conditions. We define the canonical Poisson bracket according to

$$\{j_1(x), \pi(y)\} = \delta(x-y)\ . \tag{B.8}$$

Inverting (B.7) we obtain

$$-\partial_1 \pi = j_0 \tag{B.9}$$

and hence

$$\{j_0(x), j_1(y)\} = \partial_x \delta(x-y)\ . \tag{B.10}$$

On the other hand $j_0 = \dot{\phi} = P\,, j_1 = \partial_1 \phi$ and hence using $\{\phi(x), P(y)\} = \delta(x-y)$ we obtain

$$\{j_0(x), j_1(y)\} = \{P(x), \partial_y \phi(y)\} = -\partial_y \delta(x-y) = \partial_x \delta(x-y) \tag{B.11}$$

that coincides with (B.10). The only subtlety one has to take into account is the presence of the zero modes of ϕ and π. Luckily they are finite in number and can be dealt with separately in connection with the choice of boundary conditions.

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JHEP08(2006)030